STABILITY OF THE BOUNDARY LAYER ABOVE THE SURFACE OF A WAVE TRAVELLING OVER A PLATE

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The two-dimensional problem of the stability of the flow of an incompressible fluid over a rigid surface perturbed by a wave travelling in the propagation direction of the flow is discussed in the linear approximation. The problem is solved in the coordinate system at rest with respect to the travelling wave. The parameters of this wave are not eigenvalues of the corresponding linear problem of the stability. The solution is sought in the form of a series in powers of the wave amplitude with an accuracy out to the quadratic term inclusively. Calculations are made of the dependence of the neutral stability curve on the amplitude, wavelength, and phase velocity.

1. Dimensionless quantities are used, and u_0 (the velocity of the advancing flow) and $\delta^* = 1.73\sqrt{\nu u_0/X}$, where X is the distance from the origin of the plate and ν is the kinematic viscosity, served as the scales. The discussion is conducted in the coordinate system at rest with respect to the travelling wave; therefore the coordinate of the wall has the form

$$y = \varepsilon \cos \beta x, \tag{1.1}$$

where y, x are the normal and longitudinal coordinates, respectively, ε is the amplitude, and β is the wave number. Let us introduce the following coordinates: η is the stream function and ξ is the potential of the corresponding nonviscous problem, i.e.,

$$\Delta\eta = 0;\ \eta = 0,\ y = \varepsilon \cos \beta x;\ y \to \infty,\ \partial\eta/\partial y \to 1.$$

The Lamé coefficients [1] are

$$h_\eta^{-2} = h_\xi^{-2} = h = 1 + 2\varepsilon\beta \cos \beta\xi\, e^{-\beta\eta} + O(\varepsilon\beta)^2. \tag{1.2}$$

The linearized equation for the stream function of the perturbation ψ has the form

$$\frac{\partial\Delta\psi}{\partial t} + \frac{\partial(\Psi, h\Delta\psi)}{\partial(\eta, \xi)} - \frac{\partial(h\Delta\Psi, \psi)}{\partial(\eta, \xi)} = \frac{\Delta(h\Delta\psi)}{\mathrm{Re}} \tag{1.3}$$

in the new coordinates, where t is the time and Re is the Reynolds number.

The coefficients in Eq. (1.3) are periodic functions of ξ and do not depend on t. Therefore we will seek the solution of (1.3) in the form

$$\psi = e^{pt}\varphi(\eta, \xi). \tag{1.4}$$

According to [1], the stream function of a steady flow is of the form

$$\Psi = \int (u - c)\, d\eta + \varepsilon(\Psi_{11} + \Psi_{-11}) + \varepsilon^2(\Psi_{02} + \Psi_{22} + \Psi_{-22}) + O(\varepsilon^3), \tag{1.5}$$

where the first subscript denotes the harmonic number, u is the Blasius velocity distribution, and c is the phase velocity of the wave travelling over the wall. Substituting Eq. (1.4) into Eq. (1.3), we write the problem obtained in the form

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$$L\varphi = \varepsilon(e^{-i\beta\xi}H_-\varphi + e^{i\beta\xi}H_+\varphi) + \varepsilon^2 M\varphi + O(\varepsilon^3). \tag{1.6}$$

The form of L, H_-, and H_+ is determined by Eqs. (1.2), (1.3), and (1.5). Since the eignevalues of the operator L are simple [2] the solution of Eq. (1.6) will be sought in the form [2],

$$p = \sum_0^\infty p_n \varepsilon^n, \quad \varphi = \sum_{-\infty}^{\infty} e^{in\beta}\varphi_n, \quad \varphi_n = \sum_{k=|n|}^{\infty} \varphi_{nk}\varepsilon^k. \tag{1.7}$$

Grouping terms of identical harmonics and powers of ε in (1.6), we obtain the system of equations

$$\begin{gathered} T(\alpha, p_0)\varphi_{00} = 0, \quad T(\alpha, p_0)\varphi_{01} = p_1 \operatorname{Re}\left(\frac{\partial^2}{\partial\eta^2} - \alpha^2\right)\varphi_{00}, \\ T(\alpha \mp \beta, p_0)\varphi_{\mp 11} = e^{\mp i\alpha\xi}H_{\mp}e^{\pm i\alpha\xi}\varphi_{00}, \\ T(\alpha, p_0)\varphi_{02} = p_2 \operatorname{Re}\left(\frac{\partial^2}{\partial\eta^2} - \alpha^2\right)\varphi_{00} + e^{-i(\alpha+\beta)\xi}H_- e^{i(\alpha+\beta)\xi}\varphi_{11} \\ + e^{-i(\alpha-\beta)\xi}H_+ e^{i(\alpha-\beta)\xi}\varphi_{-11} + M_0\varphi_{00}, \end{gathered} \tag{1.8}$$

where $T(\alpha, p)$ is the Orr–Sommerfeld operator. From the condition of solvability of the system (1.8) we obtain a correction to the eigenvalue with an accuracy up to ε^2

$$p_1 = 0, \quad p_2 = -\frac{1}{\operatorname{Re}}\int_0^\infty \left(e^{-i(\alpha+\beta)\xi}H_- e^{i(\alpha+\beta)\xi}\varphi_{11} + e^{-i(\alpha-\beta)\xi}H_+ e^{i(\alpha-\beta)\xi}\varphi_{-11} + M_0\varphi_{00}\right)\chi d\eta \Big/ \int_0^\infty (\varphi_{00}'' - \alpha^2\varphi_{00})\chi d\eta \tag{1.9}$$

(χ is the eigenfunction of the operator conjugate to the Orr–Sommerfeld operator).

2. We will introduce expressions for the operators entering into Eqs. (1.7),

$$\begin{gathered} H_- = \operatorname{Re}\left(p_0\Delta + \frac{\partial(\Psi_{11}, \Delta)}{\partial(\eta, \xi)} - \frac{\partial(\Delta\Psi_{11},)}{\partial(\eta, \xi)} + i\beta^2 e^{-\beta\eta}\frac{\partial\Psi_0}{\partial\eta}\Delta \right. \\ \left. + \beta^2 e^{-\beta\eta}\frac{\partial^2\Psi_0}{\partial\eta^2}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\right) - 2i\beta^2 e^{-\beta\eta}\left(\frac{\partial}{\partial\xi} - i\frac{\partial}{\partial\eta}\right), \\ M_i = \operatorname{Re}\left(\frac{\partial\Delta\Psi_{i2}}{\partial\eta}\frac{\partial}{\partial\xi} - \frac{\partial\Psi_{i2}}{\partial\eta}\frac{\partial\Delta}{\partial\xi}\right), \quad M = \Sigma M_i, \\ H_- = H_+ + \operatorname{Re}(p_0 - \bar{p}_0)\Delta, \end{gathered}$$

where

$$\Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}; \quad \Psi_0 = \int (u - c)\, d\eta; \quad \Psi_{11} = \left(F + (u - c)\frac{e^{-\beta\eta}}{2}\right)e^{i\beta\xi}.$$

Here F is the solution of the problem [1]

$$\begin{gathered} i\beta \operatorname{Re}((u - c)(F'' - \beta^2 F) - u''F) = \Delta^2 F + \frac{e^{-\beta\eta}}{2}(u^{\mathrm{IV}} - 2\beta u'''), \\ F(0) = \frac{c}{2}, \quad F'(0) = -\frac{1}{2}(u'(0) - \beta c), \quad \eta = \infty, \quad F = F' = 0. \end{gathered}$$

The principal term in Ψ_{02} is determined from the equation

$$\Delta^2\Psi_{02} = -2\beta \operatorname{Im}(\bar{F}'(F'' - \beta^2 F)) - F(\bar{F}''' - \beta^2\bar{F}')),$$

for $\eta = 0$, $\Psi_{02} = \Psi_{02}' = 0$, and for $\eta = \infty$, $\Psi_{02}'' = \Psi_{02}''' = 0$.

3. One can determine the region of applicability of the computational procedure being used by analyzing the behavior of the series (1.5). Since the calculations are performed for values of β and c close to the eigenvalue of the Orr–Sommerfeld operator, one should judge the behavior of the series from the ratio of Ψ_{13} to Ψ_{11}:

$$\Psi_{11} \sim \frac{\beta\int(u - c)u'\chi d\eta}{\Delta c\int(\partial^2/\partial\eta^2 - \beta^2)\varphi_{00}\chi d\eta} \sim \frac{\beta u'^2 y_c^2}{\Delta c}\varphi_{00},$$

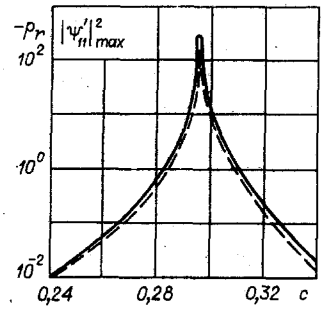

Fig. 1

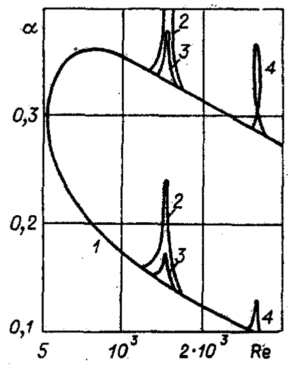

Fig. 2

$$\Psi''_{02} \sim 2\beta\,\mathrm{Re}\int_{\infty}^{\eta}\Psi'^2_{11}\,d\eta \sim \frac{2\beta^3\mathrm{Re}\,y_c^4 u'^4}{\Delta c}\int_{\infty}^{\eta}\varphi'^2_{00}\,d\eta,$$

$$\Psi_{13} \sim \varphi_{00}\int_{\infty}^{\eta}\Psi'_{11}\Psi''_{02}\chi\,d\eta/\Delta c\int_{\infty}^{\eta}\left(\frac{\partial^2}{\partial\eta^2}-\beta^2\right)\varphi_{00}\chi\,d\eta \sim \frac{2\beta^4\,\mathrm{Re}\,y_c^8 u'^6}{\Delta c^4}\varphi_{00}.$$

It was assumed in the calculations that $\varphi'_{00} \sim \chi' \sim 1$. The scale of these functions is $y_c = 2.5(\beta\,\mathrm{Re}\,u')^{-1/3}$ [3]. It follows from the estimates given that the procedure being used is suitable in the region in which the inequality

$$500\beta(\varepsilon u')^2/\Delta c^3\,\mathrm{Re} \ll 1 \tag{3.1}$$

is satisfied. Here Δc is the distance from c to the eigenvalue of the Orr–Sommerfeld operator, and β and ε are parameters of the wave (1.1). The orthogonalization method [4] was used in the numerical integration of the Orr–Sommerfeld equation. The dependence of the correction to the rise coefficient p_2 on the phase velocity of wave propagation on the wall is represented by the solid curve in Fig. 1 for a specified Tolmin–Schlichting wave, and the dependence of $|\Psi'_{11}|^2_{max}$ on c is depicted by the dashed curve. These results correspond to the lower branch of the curve of neutral stability for the values Re = 1450, $\beta = 0.141$, and $\alpha = 0.142$. As is evident from Fig. 1, the effect of wall corrugation on the stability is mainly related to a variation of the contribution to the basic steady flow. Some increase in the effect is caused by a shift of the maximum of $|\Psi'_{11}|$ into the region of the critical layer. For c = 0 the correction to the rise coefficient of the perturbation is reduced to a minimum, and $p_2 \sim 1/\mathrm{Re}$.

Similar results were obtained for the second branch of the curve of neutral stability. In this case the wall corrugation leads to destabilization of the flow. The correction to the rise coefficient in the calculations due to Ψ_{02} did not amount to more than 10% of the correction due to Ψ_{11} in the region of the neutral curve. A change in sign of the correction occurs in the region of the maximum rise coefficients of the perturbations.

The dependence of the curve of neutral stability on ε is given in Fig. 2 for the specified c and β. The neutral curve for a smooth wall is denoted by the number 1. Perturbation of the surface by a monochromatic wave leads to a distortion of the curve of neutral stability in the narrow zone where $ip_0 \approx c\beta$ and $\beta \approx \alpha$. The distortions of the neutral curve for the following parameters of a wave travelling over the wall are denoted by the numbers 2 and 3: $c = 0.296$, $\beta = 10^{-4}\,\mathrm{Re}$, and $\Delta c = 3\cdot10^{-3}$ for $\varepsilon = 10^{-3}$ and $\varepsilon = 2\cdot10^{-3}$, respectively, and by the number 4 for $c = 0.31$, $\beta = 0.9\cdot10^{-4}\,\mathrm{Re}$, $\Delta c = 10^{-2}$, and $\varepsilon = 0.7\cdot10^{-3}$. Specification of β according to this law indicates that the dimensional length of the wave travelling over the wall is preserved with a change in the number Re. The correction to the rise coefficient is appreciable only in a narrow range of Re numbers adjacent to the region of distortion of the neutral stability curve.

The results of this paper show that the effect of corrugation in the range of Reynolds numbers considered ($520 \le \mathrm{Re} \le 3500$) reduces upon satisfaction of the condition (3.1) to a distortion of the steady velocity distribution, which leads to a change in the stability of the flow. Stabilization of the flow on the lower branch of the neutral curve for $\alpha \approx \beta$ and $\Delta c \ll 1$ and destabilization on the upper branch seem natural if one recalls the results of the nonlinear theory [5, 6]. The authors are grateful to S. A. Gaponov, who pointed out this analogy.

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